Analysis

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1 Distance and Metric Spaces

In real analysis, you have encountered notions of convergence for real sequences and functions from $\mathbb{R}$ to $\mathbb{R}$. Intuitively, it seems also meaningful to say that a sequence of vectors of $\mathbb{R}^n$ tends toward a limiting vector, that a sequence of functions tends toward a limiting function, or that any object can possibly tend toward a limiting object. We want a more general framework in which such notions of convergence and limits—and others to be defined—will be precisely defined. We first want to make the minimal assumptions on a set for these notions to be meaningful. Fundamentally, we need only one assumption: that there exists a way to measure the distance between two points in the set. So first we need to give a precise meaning to the intuitive idea of distance.

Definition 1.1. *Let X be a set. A* ***distance*** *or* ***metric*** *on X is a function $d : X^2 \mapsto \mathbb{R}+$ (*positive *reals) that satisfies the 3 following axioms:*

1. *$d(x, y) = 0$ iff $x = y$.*
2. *Symmetry: $d(x, y) = d(y, x)$.*
3. *Triangle inequality: for all z, $d(x, y) \leq d(x, z) + d(z, y)$.*

A set X together with a metric d, i.e. the couple (X, d), is called a ***metric space****.*
Elements of a metric space are called ***points****.*

Defining and noting a metric space as the couple (X, d) instead of just X may seem cumbersome, but it is important to keep track that the distance defined on the space X is part of what the metric space is: if there are two distances d_1 and d_2 defined on X, there are two metric spaces (X, d_1) and (X, d_2). Properties that apply to (X, d_1) may well not apply to (X, d_2). Only when there is no ambiguity about what the distance is can we just designate the metric space as X.

Given a metric space X endowed with the distance d, any subset S endowed with the same distance is a metric space too. (Rigorously, it is not exactly the same function d since the domain of it is now S, but we still note it d).

On $\mathbb{R}$, we are used to using the absolute value of the difference of two reals $|x-y|$ as a distance. On $\mathbb{R}^n$, we are used to using $d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = ||x-y||_2$. This are particular cases of a more general case:

Proposition 1.1. *Let $(V, ||.||)$ be a normed vector space. Then V is a metric space if endowed with the metric defined by:*

$$d(x,y) = ||x-y||$$

This is straightforward to see going from the definition axioms of a norm to the definition axioms of a distance. Because we can turn any inner-product space into a normed vector space by defining $||x|| = \sqrt{< x, x >}$, it follows that we can make any inner-product space a metric space by defining $d(x,y) = \sqrt{< x-y, x-y >}$. If the inner-product is finite-dimensional—an Euclidian space—this distance is then called the **Euclidian distance**. On $\mathbb{R}^n$, the Euclidian distance associated to the dot product (and hence to the 2-norm $||.||_2$) is the distance $d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = ||x-y||_2$ mentioned above.

Once we have a distance, we can meaningfully tell if a given subspace if bounded or unbounded:

Definition 1.2. *A subset S of a metric space (X,d) is **bounded** iff there exist a point $a \in X$ and a real number M such that for all $x \in S$, $d(x,a) \leq M$.*

Note that the point a does not matter much: if there exists a such that for all $x \in X$, $d(x,a) \leq M$, then for any a', there exists an M' such that for all $x \in X$, $d(x,a') \leq M'$. This is just an application of the triangle inequality: $d(a',x) \leq d(a',a) + d(a,x) \leq d(a',a) + M$. Choosing $M' = M + d(a,a')$ works. For a normed vector space V, we usually pick $a = 0$ to prove that a subset S if bounded: S is bounded iff there exists M such that for all $x \in S$, $||x|| \leq M$.

2 Open and closed sets, interior and closure

Before defining convergence, we define the interior and closure of a set. To see where we are going, come back to the real line that we are aiming at generalizing. Consider for instance the open interval $(0, 1)$. Any point in $\mathbb{R}$ either belongs or does not belong to $(0, 1)$: there is no in-between. However, consider 0 and 1: although they do not belong to $(0, 1)$, we would like to express the idea that, well, you know, they almost do. They really seem to be closer to belonging to $(0, 1)$ than, say, the number 9. Or consider the closed interval $[0, 1]$. This time, we would like to say that even though 0 and 1 belong to $[0, 1]$, it's like, they almost don't. They really seem to belong less to $[0, 1]$ than, say, $1/2$, which is really inside $[0, 1]$. Put otherwise, we would like to be able to say that, although belonging to a subset is a binary notion, there are points that are really inside a subset, others that are really outside the subset, and some—which may or may not belong to the subset—that are on the boundary of the subset. In the example of intervals of $\mathbb{R}$, we can define the boundary points as 0 an 1, but what for other subsets of $\mathbb{R}$, or subsets in more general metric spaces? In this subsection, we aim at making precise these intuitive notions.

2.1 Open balls and neighborhoods

Once we have a distance—a way to assess how far two elements are from one another—we can talk about the neighborhood of a point.

Definition 2.1. *An* ***open ball*** *with* ***center*** *a and* ***radius*** *$r > 0$ is the subset of all points of X at a distance less than r of a:*

$$B(a, r) = \{x \in X / d(x, a) < r\}$$

On $\mathbb{R}$, open balls are open intervals $(a - r, a + r)$.

Definition 2.2. *A* ***neighborhood*** *N of a point a is a subset of X containing an open ball of center a: $a \in B(a, r) \subseteq N$.*

(Be careful that some authors (Rudin) use the term neighborhood to refer to an open ball).

2.2 Interior point, interior, and open sets

First, we want to strengthen the notion of belonging to a set: we want to distinguish among points that belong to a subset S between those that are in the inside, and those that are on the edge. The idea of being in the inside—being interior—is formalized by saying that when moving a bit away from the point—this can be very little—we are still in the set.

Definition 2.3. *Let (X, d) be a metric space and S a subset of X.*

- *A point $a \in X$ is an* ***interior point*** *of S iff there exists a radius $r > 0$ such that the open ball $B(a, r)$ is included in S:*

$$\exists r > 0 / a \in B(a, r) \subseteq S$$

- *The* ***interior*** *of S, noted $\mathring{S}$ or $int(S)$ is the set of all its interior point.*
- *Because an interior point of S always belong to S, $int(S) \subseteq S$.*
- *A subset S is an* ***open set*** *iff all its points are interior points of S, i.e. iff it is equal to its interior $int(S) = S$.*

The empty set is an open set (by definition, although vacuously). The whole initial set X is an open set. The interior of any set is an open set (you are asked to prove it in the problem set). Also:

Proposition 2.1. *Any open-ball is an open set.*

Proof. Nope, the terminology is no proof; we have to prove the result! Let $x \in B(a, r)$. The only thing to do is to pick a radius $R > 0$ for a ball around x, and show that any $y \in B(x, R)$ belongs to $B(a, r)$. Pick $R = r - d(a, x) > 0$, so that $d(x, y) < r - d(a, x)$. Using the triangle inequality, $d(y, a) \leq d(y, x) + d(x, a) < r$. QED. □

Theorem 2.1.

1. *Any union (possibly infinite) of open sets is an open set.*
2. *Any* finite *intersection of open sets is an open set.*

Proof. Let $(S_i), i \in I$ be a family of open sets, and consider their union $S = \cup_i S_i$. Let $x \in S$; x belongs to an S_i for some i. Since S_i is open, there exists r such that $x \in B(x, r) \subseteq S_i$. But $S_i \subseteq S$, so $x \in B(x, r) \subseteq S$, which proves that x is an interior point of S too.

Let $(S_i)_{i=1}^n$ ba a finite family of open sets, and consider their intersection $\cap_i S_{i=1}^n$. Let $x \in S$. Since x belongs to all S_i, for all i there exists r_i such that $B(x, r_i) \subseteq S_i$. Pick $R = \min(r_1, ..., r_n)$; $B(x, R)$ is included in all open balls $B(x, r_i)$, hence in all S_i, hence in S.

□

Be careful that in general an infinite intersection of open sets may not be an open set. Consider for instance the infinite family of sets in $\mathbb{R}$ (endowed with the absolute value) $S_n = (-1/n, 1/n)$ for all n. They are open sets (since they are open balls). But $\cap_{n\in\mathbb{N}} S_n = \{0\}$, which is not an open set of $\mathbb{R}$ (check it!).

2.3 Closure points, closure and closed sets

Second, we want to weaken the notion of belonging to a set: we want to be able to say that in addition to the points that belong to S, some points outside of S may be so close that they can be considered to be on the boundary of S. The idea of a point x being close to a set S is formalized by saying that however little we depart from x, we always encounter points of S.

Definition 2.4. *Let (X, d) be a metric space and S a subset of X.*

- *A point $a \in X$ is a **closure point** of S iff for any radius $r > 0$, the open ball $B(a, r)$ around a contains some point of S:*

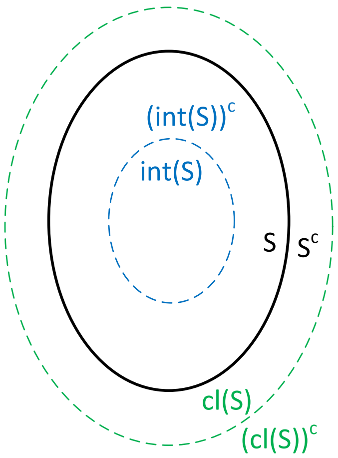
$$\forall r > 0, B(a, r) \cap S \neq \emptyset$$

- *The **closure** of a subset S, noted $\bar{S}$ or $cl(S)$, it the set of all its closure points.*
- *Because a point that belongs to S is always a closure point of S, $S \subseteq cl(S)$.*
- *A subset S is a **closed set** iff its only closure points are the points that belong to S—iff its closure reduces to itself $cl(S) = S$.*

The empty set is a closed set (so that it is both open and closed). The whole set X is a closed set (so that it is both open and closed). (Let us get rid of a confusion that may arise from the vocabulary: contrary to a door, a subset can be both open and closed, and it can be neither). The closure of any set is a closed set.

To recap, for any set S, we have defined a subset of S—the interior of S, and a set that contains S—the closure of S:

$$int(S) \subseteq S \subseteq cl(S)$$



If you are into potatoes drawings, consider a subset S and take a look at the figure above. We can draw a solid line that delimits S and its complement S^c. Now we can draw doted lines that represented the delimitations between $int(S)$ and its complement, and $cl(S)$ and its complement. Informally speaking, we can think of them as three delimitations between S and S^c: the true one, one that (in general) puts more points in S, and one that (in general) puts fewer points in S. What if we change perspectives, that is if we consider the set S^c? The true delimitation between S^c and $(S^c)^c = S$ is the one between S and S^c (the solid line.) But what about $int(S^c)$ v. $(int(S^c))^c$ and $cl(S^c)$ v. $(cl(S^c))^c$? Do they create additional delimitations? Nope, because:

Proposition 2.2.

- *The complement of $int(S)$ is the closure of S^c: $(int(S))^c = cl(S^c)$.*
- *The complement of $cl(S)$ is the interior of S^c: $(cl(S))^c = int(S^c)$.*

Proof. The definition of belonging to $int(S)$ is:

$$a \in int(S) \Leftrightarrow \exists r > 0 / B(a,r) \subseteq S$$

Take the negation of each side of the equivalence:

$$a \in (int(S))^c \Leftrightarrow \forall r > 0 / B(a,r) \cap S^c \neq \emptyset$$

But the right-hand side is precisely the definition of belonging to $cl(S^c)$! The second result is then just the first one applied to S^c: $(int(S^c))^c = cl(S)$. □

An important theorem directly follows: $int(S) = S$ iff $cl(S^c) = S^c$, and $cl(S) = S$ iff $int(S^c) = S^c$. Or in words:

Theorem 2.2.

- *A subset S is an open set iff its complement S^c is a closed set.*
- *A subset S is a closed set iff its complement S^c is a open set.*

A consequence of this theorem is that we can deduce from theorem 2.1 that:

Theorem 2.3.

1. *Any intersection (possibly infinite) of closed sets is a closed set.*
2. *Any finite union of closed sets is a closed set.*

Proof. Simply use Morgan's laws $\cap_i S_i = (\cup_i S_i^c)^c$ and $\cup_{i=1}^n S_i = (\cap_{i=1}^n S_i^c)^c$ and use theorem 2.1 along with the previous theorem. □

2.4 The boundary

So look again at the drawing of potatoes. Around the true delimitation between S and S^c, we have defined two delimitations, which create a partition of X in three sets: first the interior of S, second the complement of the closure of S, which it the interior of S^c, and finally a set in between. We call this in-between the boundary of S.

Definition 2.5. *The* ***boundary*** *of a subset S, noted ∂S is the set difference between its closure and its interior: $\partial S = cl(S) - int(S) = cl(S) \cap (int(S))^c$.*

Without surprise, the boundary of S is also the boundary of S^c: $\partial S = cl(S) \cap (int(S))^c = (int(S^c))^c \cap cl(S^c)$. To sum up, for any subset S of X, we can make the following partition of X:

1. The interior of S, $int(S)$. If $a \in int(S)$, then for all $r > 0$ smaller than a certain value, the open ball $B(a, r)$ contains only elements of S.
2. The boundary of S, ∂S. The boundary can be divided between:
   - The boundary that belong to S, $\partial S \cap S$. If $a \in \partial S \cap S$, then for all $r > 0$, because $a \in cl(S)$, the open ball $B(a, r)$ contains an element of S^c. But $B(a, r)$ also contains an elements of S: a.

- The boundary that belong to S^c, $\partial S \cap S^c$. If $a \in \partial S \cap S^c$, then for all $r > 0$, because $a \in cl(S^c)$, the open ball $B(a, r)$ contains an element of S. But $B(a, r)$ also contains an elements of S^c: a.

Therefore, if $a \in \partial S$, then however small r is, the open ball $B(a, r)$ always contains both at least one element of S and one element of S^c.

3. Points that do not belong to the closure of S, $(\bar{S})^c$. It is the interior of S^c, $(int(S^c)$. If $a \in int(S^c)$, then for all r smaller than a certain value, the open ball $B(a, r)$ contains only elements of S^c.

Not all parts in the partition are always non-empty. A subset can have an interior reduced to the empty set; a subset can have the whole space X as its closure; when a set is closed and open, $cl(S) = int(S)$, its boundary is empty.

3 Sequences and Convergence

3.1 Definitions and Basic Properties

Sequences are defined in any set X,

Definition 3.1. *A* ***sequence*** *in a set X is a function from $\mathbb{N}$ to X. We note x_n the image of n, and (x_n) the sequence. By analogy with the cartesian product, we note $X^{\mathbb{N}}$ the set of sequences on X.*

but if a space X is a metric space, we can define properties of sequences that rely on the notion of distance. First, boundedness:

Definition 3.2. *Let (X, d) be a metric space. A sequence (x_n) of X is* ***bounded*** *iff its image $\{x_1, ..., x_n, ...\}$ is bounded, i.e. iff there exists a point a and a real number M such that for all n, $d(x_n, a) \leq M$.*

More importantly, if X is a metric space, then we can inquire whether a sequence of X has a limit.

Definition 3.3. *Let (X, d) be a metric space. A sequence (x_n) of X is said to* ***converge in X to a limit*** *$l \in X$ iff for all positive real ε, there exists an integer $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$, $d(x_n, l) < \varepsilon$:*

$$\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N} / \forall n \geq N(\varepsilon), d(x_n, l) < \varepsilon$$

We write $x_n \to l$ or $\lim_{n\to\infty} x_n = l$.

The notation $N(\varepsilon)$ is here to stress that N can be different for each ε. Note that:

- $d(x_n, l) < \varepsilon$ means that $x_n \in B(l, \varepsilon)$. So in the words of the previous section, a sequence (x_n) converges to l iff, however small the radius of an open ball around l, all elements of (x_n) belong to this open ball after a certain rank $N(\varepsilon)$.
- The distance used is part of the definition of a metric space: a sequence may converge in (X, d_1) but not in (X, d_2)
- The seemingly innocuous requirement that $l \in X$. Consider a sequence (x_n) of X such that all terms belong to a subset S of X. We can look at (x_n) either as a sequence of X or a sequence of S. But if (x_n) converges in X, it follows that (x_n) converges in S only if $l \in S$. It is therefore important, when there is a possible ambiguity, to precise in which set a sequence converges.

Some properties of convergent sequences now.

Proposition 3.1. *If a sequence converges, then it is bounded.*

Proof. Assume $x_n \to l$. Pick for instance $\varepsilon = 1$ and use the definition of convergence: there exists an integer N such that for all $n \geq N$, $d(x_n, l) < 1$. This proves that the sequence starting at N is bounded, but we need to deal with the $N-1$ first terms $x_1, ...x_{N-1}$. If we define $M = \max(d(x_1, l), ..., d(x_{N-1}, l), 1)$, then for all n, $d(x_n, l) < M$. QED. □

Proposition 3.2. *The limit of a sequence if unique provided it exists.*

Proof. You are asked to prove it in the problem-set. □

Although most of the results we derive are concerned with any metric spaces, here is one that is specific to $\mathbb{R}^n$, endowed with the Euclidian distance.

Proposition 3.3. *A sequence $(x^k) = (x_1^k, ..., x_n^k)$ of $\mathbb{R}^n$ endowed with the Euclidian distance converges to a limit l iff each component converges to the corresponding component of l in $\mathbb{R}$ endowed with the absolute value metric.*

Proof. The whole proof relies on two observations (one to prove each implications of the equivalence). First, since $||x||_2 = \sqrt{x_1^2 + ... + x_n^2}$, $|x_i| \leq ||x||_2$ for all $i = 1...n$; second, for all $i = 1...n$, $||x||_2 \leq \sqrt{n} \max_i(|x_i|)$.

Assume the sequence (x^k) converges to a limit l. Fix a component i. Fix ε; there exists an integer K such that for all $k \geq K$, $||x^k - l||_2 < \varepsilon$. So $|x_i^k - l_i| \leq ||x^k - l||_2 < \varepsilon$.

Conversely, assume that for all $i = 1...n$, (x_i^k) converges to a limit l_i. Fix ε; for each i, there exists an integer K_i such that for all $k \geq K_i$, $|x_i^k - l_i| < \varepsilon/\sqrt{n}$. Define $l = (l_1, ..., l_n)$. Define $K = \max(K_1, ..., K_n)$ and consider $k \geq K$. For all $i = 1, ..., n$, $|x_i^k - l_i| < \varepsilon/\sqrt{n}$, so $\max_{i=1}^n(|x_i^k - l_i|) < \varepsilon/\sqrt{n}$. Hence, $||x^k - l||_2 \leq \sqrt{n} \max_i(|x_i^k - l_i|) < \varepsilon$. □

3.2 Operations on limits

So far, we have only assumed that X is a metric space; no operation is defined on X. In a vector space where we have an addition and a scalar multiplication, we have the following results, if we use a distance induced by a norm:

Proposition 3.4. *Let $(V, ||.||)$ be a normed vector space and consider the distance induced by the norm $||.||$. Let (x_n) and (y_n) be sequences of V, and $\lambda \in \mathbb{R}$. If $x_n \to x$ and $y_n \to y$, then:*

- $x_n + y_n \to x + y$

- $\lambda x_n \to \lambda x$

In $\mathbb{R}$ (endowed with the absolute value metric), if (x_n) and (y_n) are sequences that converge to x and y, then:

- $x_n y_n \to xy$
- $1/x_n \to 1/x$ *if* $x \neq 0$

Proof. The proofs are left as exercises. □

3.3 Sequential characterization of the closure of a subset

The closure can be characterized sequentially: it is the points that can be approximated by a sequence of S:

Proposition 3.5. *Let S be a subset of X. A point x is a closure point of S iff there exists a sequence of elements of S, $(x_n) \in S^{\mathbb{N}}$, that converges (in X) to x.*

Note that here, it is important that (x_n) converges in X, not S: (x_n) never converges in the metric space S if $x \notin S$.

Proof. Assume x is a closure point of S. We are going to construct of sequence of S that converges to x, that is, to each n, we are going to associate an element of S, then check that the constructed sequence converges to x. The idea is to pick x_n closer and closer to x as n increases. Let $n \in \mathbb{N}$; because x is a closure point of S, the open ball around x with radius $1/n$ $B(x, 1/n)$ intersects S, so there exists an element of S that belongs to S; we pick it to define x_n. Now let us check that the constructed sequence converges. Take $\varepsilon > 0$. There exists an integer N such that $1/N < \varepsilon$. Consider an $n \geq N$, it belongs to $B(x, 1/n)$ so $d(x_n, x) < 1/n \leq 1/N < \varepsilon$. QED.

Conversely, assume there exists $(x_n) \in S^{\mathbb{N}}$ that converges to x. Take an open ball with radius r around x, $B(x, r)$. We want to show it intersects S. But choosing $\varepsilon = r$ in the definition of convergence, we know that there exists N such that x_N belongs to S (we even know that all subsequent terms of the sequence belongs to S!). □

A particular case of this result concerns the real line: if (x_n) is a sequence of reals such that $a < x_n < b$ for all n, and (x_n) converges to a limit l, then $a \leq l \leq b$. This is because the closure of the open interval $]a, b[$ is the closed interval $[a, b]$.

Another way to phrase the proposition is to say that the closure of S is the set of limits of convergent sequences of S. So:

Corollary 3.1. *S is close iff the limits of all convergent sequences of S belong to S.*

4 Compactness

4.1 Definition(s)

There are two definitions of compactness. Although they are equivalent (in the framework of metric spaces that we consider), they have two different names—compactness and sequential compactness—for reasons detailed below.

4.1.1 Definition with Subsequences

The first definition relies on the notion of subsequences. Consider a sequence (x_n). A subsequence is a sequence obtained from (x_n) by *"dropping some terms"*. For instance we could define a subsequence (y_k) of (x_n) by defining $y_1 = x_3$, $y_2, = x_5$, $y_3 = x_6$, $y_4 = x_9$, etc.—that is by dropping x_1, x_2, x_4, x_7, x_8, etc. The key in the definition is that we need to respect the order in which terms appear in (x_n): we cannot *"go back"* and add $y_5 = x_7$ after y_4. This means that the subsequence (y_k) is obtained from (x_n) by defining a *strictly increasing* function from $\mathbb{N}$ to $\mathbb{N}$ which to each integer k associates which term n_k of (x_n) is chosen to be y_n.

Definition 4.1. *Let (x_n) be a sequence. A **subsequence** of (x_n) is a sequence (x_{n_k}), where the function $k \mapsto n_k$ is a strictly increasing function from $\mathbb{N}$ to $\mathbb{N}$.*

Note that:

Proposition 4.1. *A sequence (x_n) converges to a limit l iff all its subsequences converge to the same limit l.*

Proof. Since (x_n) is a subsequence of itself, one implication is obvious. We prove the converse; assume (x_n) converges to a limit l and consider a subsequence (x_{n_k}) of (x_n). Fix $\varepsilon > 0$. Because (x_n) converges, there exists an integer N such that for all $n \geq N$, $d(x_n, l) < \varepsilon$. Take K such that $n_K \geq N$. Then for all $k \geq K$, $n_k \geq n_K \geq N$, so $d(x_{n_k}, l) < \varepsilon$. QED. □

However, having *one* converging subsequence does not imply convergence: *"having a convergent subsequence"* is a weaker requirement than converging. That a sequence (x_n) converges to a limit l means that as n increases to infinity, all terms are arbitrary close to l. That a *sub*sequence of (x_n) converges to l only means that as n increases to infinity, we will always find terms that are arbitrarily close to l—the terms that belong to the subsequence. We are now ready to define a compact space through the sequential definition.

Definition 4.2. *A metric space* (X, d) *is* ***sequentially compact*** *iff every sequence of* X *has a converging subsequence.*

A subset S *of a metric space* (X, d) *is* ***sequentially compact*** *iff every sequence of* S *has a subsequence that converges to a limit* $l \in S$.

Intuitively, a compact space is *"compact"* indeed: there is little room in it, so that for any sequence, there is at least one point near which the sequence always comes back (or stays).

4.1.2 Definition with Open Coverings

The second definition relies on the notion of open coverings and subcoverings.

Definition 4.3. *Let* (X, d) *be a metric space.*

- *An* ***open covering*** *of a subset* S *is a collection of open sets* $(S_i)_i$ *of* X *whose union contains (covers)* S*:* $\cup_i S_i \supseteq S$.
- *A* ***subcovering*** *of an open covering* $(S_i)_i$ *is obtained by deleting some of the sets in such a way that the surviving ones still cover* S*—that they still form an open covering.*

Note that any set X has an open covering: just take a union of open balls centered in each element of the set, for instance $\bigcup_{x \in X} B(x, 1)$. A compact set requires more than that:

Definition 4.4. *A subset* S *of a metric space* (X, d) *is* ***compact*** *iff* every *open covering of* S *admits a finite subcovering.*

4.1.3 Equivalence between the two notions of compactness

Theorem 4.1. *Compactness is equivalent to sequential compactness (in metric spaces).*

Proof. We *admit* the theorem. □

If you are curious: why the two names for the same notion then? It is possible to define the notion of a topological space, a generalization of the notion of a metric space that does not require the existence of a metric. And, in a topological space, it is possible to generalize the notion of open sets and convergence without using the notion of a metric. Compactness and sequential compactness are then defined in topological spaces in the same way as we have seen. Now it turns out that in general topological spaces, compactness and sequential compactness are not in general equivalent. They are so only in metric spaces.

4.2 Properties

First, compactness and boundedness:

Proposition 4.2. *Let S be a subset of a metric space (X, d).*

If S is compact, then it is bounded.

Proof. Let us use the sequential definition of compactness. We show the contrapositive. Assume S is not bounded. We build a sequence that has no converging subsequence. Take any element of S as x_0. Fix $n \geq 1$. Since S is not bounded, there exists an element $x \in S$ such that $d(x_0, x) > n$; define this x as x_n. Hence for all n, $d(x_0, x_n) > n$. The sequence we have built has no bounded subsequence. So it has no converging subsequence. So X is not compact. □

Second, compactness and closeness.

Proposition 4.3. *Let S be a subset of a metric space (X, d).*

If S is compact, then it is closed (in X).

Proof. We want to show that $cl(S) = S$ (in the metric space (X, d)). Let $x \in cl(S)$; we want to show $x \in S$. We use the sequential definition of compactness, and the sequential characterization of closeness. Since $x \in cl(S)$, it is the limit in X of a sequence $(x_n) \in S^{\mathbb{N}}$. Since S is compact, (x_n) has a subsequence (x_{n_k}) that converges to a limit $l \in S$. But (x_{n_k}) is a subsequence of the converging sequence (x_n), so l is necessarily equal to x. So $x = l \in S$. □

Proposition 4.4. *Let S be a subset of a metric space (X, d). Assume (X, d) is compact. Then:*

S is compact iff S is closed (in X).

Proof. We have already shown that if S is compact, then it is closed (regardless of whether X is compact or not); we prove the converse. Assume S is closed in X. Let (x_n) be a sequence of S. Since (x_n) is a sequence of X too, and X is compact, it has a subsequence (x_{n_k}) that converges to a limit $l \in X$. But since (x_{n_k}) is a sequence of S and S is closed, $l \in S$, so (x_{n_k}) converges in S. QED. □

4.3 The Bolzano-Weierstrass and Heine-Borel theorems in $\mathbb{R}^n$

From propositions 4.2 and 4.3, we know that if a subset S of a metric space (X, d) is compact, then it is closed (in X) and it is bounded. In general, the converse is not true: a bounded and closed subset of a metric space is not necessarily compact. However, it is true in $\mathbb{R}$ and $\mathbb{R}^n$, a result known as the Bolzano-Weierstrass theorem.

We prove this result in this section, through intermediary results that are of interest in themselves. For all of this section, we are considering the metric space $\mathbb{R}$ endowed with the absolute value metric, and the metric space $\mathbb{R}^n$ endowed with the Euclidian distance. A first important result:

Theorem 4.2. ***Monotone convergence theorem***

- *Every increasing and bounded from above sequence of* $\mathbb{R}$ *converges (to* $sup(x_n)$*).*
- *Every decreasing and bounded from below sequence of* $\mathbb{R}$ *converges (to* $inf(x_n)$*).*
- *Every monotone and bounded sequence of* $\mathbb{R}$ *converges.*

Proof. The proof of the first item uses the least upper-bound property in $\mathbb{R}$. Since $\{x_1, ..., x_n, ...\}$ is bounded from above and non-empty, it has a least upper bound $Sup(x_n)$. Fix $\varepsilon > 0$. $Sup(x_n) - \varepsilon$ is not an upper-bound of $\{x_1, ..., x_n, ...\}$ so there exists an element x_N such that $Sup(x_n) - \varepsilon < x_N$. Let $n \geq N$; since (x_n) is increasing, $x_N \leq x_n$, so $Sup(x_n) - \varepsilon < x_n \leq Sup(x_n)$. Hence $|x_n - Sup(x_n)| < \varepsilon$. QED. The second item of the theorem follows since if (x_n) is decreasing and bounded from below, $(-x_n)$ is increasing and bounded from above. The last item is a direct consequence of the first two. ☐

Besides, we have the result:

Lemma 4.1. *Every sequence of* $\mathbb{R}$ *has a monotonic subsequence.*

Proof. Consider a sequence (x_n) of $\mathbb{R}$. Consider the set S of all the x_n such that x_n is greater than all its subsequent terms: $\forall m > n, x_n > x_m$. There are two cases. If S is infinite, the elements of S form a (strictly) decreasing subsequence and we are done. If S is finite, we show that we can build an increasing subsequence. Let N be the greatest integer n such that $x_n \in S$, so that for all $n > N$, $x_n \notin S$. Define $n_1 = N + 1$. Since $n_1 \notin S$, there exists $m > n_1$ such that $x_m \geq x_n$. Define this m as n_2. By induction, we can continue to build an increasing sequence. ☐

Let (x_n) be a bounded sequence of $\mathbb{R}$. As any sequence, it has a monotonic subsequence; this monotonic subsequence is bounded, so it converges; therefore, (x_n) has a converging subsequence. We have just proved the Bolzano-Weierstrass theorem in $\mathbb{R}$.

Theorem 4.3. ***Bolzano-Weierstrass theorem in*** $\mathbb{R}$

Every bounded sequence of $\mathbb{R}$ *(endowed with the absolute value distance) has a converging subsequence.*

The result extends easily to $\mathbb{R}^n$, endowed with the Euclidian metric.

Corollary 4.1. ***Bolzano-Weierstrass theorem in*** $\mathbb{R}^n$

Every bounded sequence of $\mathbb{R}^n$ *(endowed with the Euclidian distance) has a converging subsequence.*

Proof. The proof is by induction on n (the base case for $n = 1$ is the Bolzano-Weierstrass theorem in $\mathbb{R}$). Let us prove for instance that $\mathbb{R}^2$ is complete.

Let $(x_k)_k = (x_k^1, x_k^2)_k$ be a bounded sequence of $\mathbb{R}^2$. Consider $(x_k^1)_k$ the first component of $(x_k)_k$. It is a sequence of $\mathbb{R}$, and since $(x_k)_k$ is bounded, it is bounded. So by Bolzano-Weierstrass theorem in $\mathbb{R}$, it has a subsequence $(x_{k_p}^1)_p$ that converges to a limit l^1.

Consider the subsequence $(x_{k_p})_p = (x_{k_p}^1, x_{k_p}^2)_p$ of $(x_k)_k$. Consider $(x_{k_p}^2)_p$ the second component of the <u>sub</u>sequence $(x_{k_p})_p$. It is a bounded sequence of $\mathbb{R}$, so it has a subsequence $(x_{k_{p_q}}^2)_q$ that converges to a limit l^2. A subsequence of a subsequence is a subsequence of the original sequence, so $(x_{k_{p_q}}^2)_q$ is a subsequence of $(x_k^2)_k$. To conclude, we show that the subsequence $(x_{k_{p_q}})_q = (x_{k_{p_q}}^1, x_{k_{p_q}}^2)_q$ of $(x_k)_k$ converges to $l = (l^1, l^2)$. We know it does if and only if we have convergence component by component. We know that $x_{k_{p_q}}^2 \to l_2$. And $x_{k_{p_q}}^1 \to l^1$ because it is a subsequence of $(x_{k_p}^1)$, which converges to l^1. □

Using the Bolzano-Weierstrass theorem, the Heine-Borel theorem is a one-line proof: consider a bounded and closed subset S of $\mathbb{R}^n$. Let (x_n) be a sequence of S. Because S is bounded, so is (x_n). So (x_n) has a converging subsequence in $\mathbb{R}^n$. But since S is close, the limit belongs to S. So S is compact.

Theorem 4.4. ***Heine-Borel theorem***

A subset of $\mathbb{R}^n$ *(endowed with the Euclidian distance) is compact iff it is bounded and closed.*

Just a remark: you may find that Heine and Borel got it easy, since deriving the Heine-Borel theorem from the Bolzano-Weierstrass theorem is not overwhelmingly complicated. Their proof however was derived independently, relying on the definition of compactness with open coverings.

5 Cauchy Sequences and Completeness

5.1 Cauchy sequences

When a sequence converges, its elements become closer and closer to the limit as n increases. As a byproduct, all terms become closer and closer together. The notion of a Cauchy sequence weakens the requirement of convergence by only requiring that the terms get closer together, without necessarily converging to a limit.

Definition 5.1. *A sequence (x_n) is Cauchy iff:*

$$\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N} / \forall n, m \geq N(\varepsilon), d(x_n, x_m) < \varepsilon$$

Since a Cauchy sequence has its terms getting closer together, it is not surprising that it is bounded, just as a convergent sequence is.

Proposition 5.1. *If a sequence is Cauchy, then it is bounded.*

Proof. The proof parallels closely the one for convergence. Pick $\varepsilon = 1$. There exists an integer N such that for all $n \geq N$, $d(x_n, x_N) < 1$. This proves that the sequence starting at N is bounded. Defining $M = \max(d(x_1, x_N), ..., d(x_{N-1}, x_N), 1)$, then for all n, $d(x_n, x_N) \leq M$. QED. □

As we said, the notion of a Cauchy sequence weakens the notion of convergence:

Proposition 5.2. *If a sequence converges, then it is Cauchy.*

Proof. Suppose (x_n) converges to a limit l. Let $\varepsilon > 0$. By the definition of convergence (applied to $\varepsilon/2$), there exists an integer N such that for all $n \geq N$, $d(x_n, l) < \varepsilon/2$. Let $n, m \geq N$. Using the triangle inequality, $d(x_n, x_m) \leq d(x_n, l) + d(l, x_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. QED. □

5.2 Completeness

But is Cauchiness really a generalization of convergence, or are all Cauchy sequences convergent? As you can guess, if we defined the concept, the answer is no in general. However, in many metric spaces, the two notions are equivalent. Such metric spaces are called complete.

Definition 5.2. *A metric space (X, d) is* ***complete*** *iff all Cauchy sequences in X converge (in X), i.e.*

iff convergence and Cauchiness are equivalent properties in X.

A complete normed vector space is called a ***Banach space****.*

Do not think that, because it is equivalent to convergence, the notion of Cauchiness becomes useless in a complete metric space. To the contrary! It becomes a very useful characterization of convergence. Why so useful? Because in using the definition of convergence, we need to first find a guess limit l to the sequence, and then show the sequence converges to l. In contrast, showing that a sequence is Cauchy in a complete metric space is a non constructive proof that it converges: we do not need to know the limit to show it exists.

Some results about completeness and closeness. Note that the notion of completeness is defined for whole metric spaces only, not their subsets. However, given a subset S of a metric space (X, d), we can always consider the metric space (S, d). As any metric space, (S, d) is closed in itself. But if (S, d) is complete, then S is closed is X:

Proposition 5.3. *Let S be a subset of a metric space (X, d).*

If S is complete (if (S, d) is a complete metric space), then it is closed in X.

Proof. Assume S is complete. Use the sequential characterization of closeness to show that $cl(S) \subseteq S$. Let $x \in cl(S)$. Let (x_n) be a sequence of S that converges to x in X. We need to show that $x \in S$. Since (x_n) converges in X, it is Cauchy. Since it is a Cauchy sequence of S complete, it converges in S, ie to a limit $l \in S$. But since l is also a limit of (x_n) in X, $l = x$ because the limit is unique. This shows that $x \in S$. □

Proposition 5.4. *Let S be a subset of a metric space (X, d). Assume (X, d) is complete. Then:*

S is complete ((S, d) is a complete metric space) iff S is closed in X.

Proof. We have already shown that if S is complete, then it is closed (regardless of whether X is complete or not); we prove the converse. Assume S is closed in X. Let (x_n) be a Cauchy sequence of S; we want to show it converges in S. It is also a sequence of X, so a Cauchy sequence of X. So it converges *in* X, i.e. to a limit $l \in X$. But since S is closed, $l \in S$, so (x_n) converges *in* S. □

5.3 Metric Spaces that are complete

Which metric spaces are complete? In this subsection, we show that any compact space, as well as $\mathbb{R}$ and $\mathbb{R}^n$ are complete. The proofs of both results rely on the following lemma:

Lemma 5.1. *If a sequence (x_n) is Cauchy and has a subsequence that converges to a limit l, then (x_n) converges to l.*

Proof. Let us note (x_{n_k}) the converging subsequence. Fix $\varepsilon > 0$. Because (x_n) is Cauchy, there exists an integer N such that for all $n, m \geq N$, $d(x_n, x_m) < \varepsilon/2$. Because (x_{n_k}) converges to l, there exists an integer K such that for all $k \geq K$, $d(x_{n_k}, l) < \varepsilon/2$. Let K' be such that both $K' \geq K$ (so that $n_{K'} \geq n_K$) and $n_{K'} \geq N$. Take $n \geq n_{K'}$. Since $n, n_{K'} \geq N$, $d(x_n, x_{n'_K}) < \varepsilon/2$. Since $K' \geq K$, $d(x_{n'_K}, l) < \varepsilon/2$. Hence, using the triangle inequality, $d(x_n, l) \leq d(x_n, x_{n_{K'}}) + d(x_{n_{K'}}, l) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. QED. □

So to prove that a space is complete, if it enough to show that a Cauchy sequence has a converging *sub*-sequence. First, compact metric spaces. Consider a Cauchy sequence (x_n) in a compact space. As a sequence of a compact space, (x_n) has a converging subsequence; from the previous lemma, (x_n) converges. In other words:

Proposition 5.5. *If a metric space is compact, then it is complete.*

Second, $\mathbb{R}^n$. Since $\mathbb{R}^n$ is not bounded, it is not compact, and we cannot use the previous proposition. However, we can use the lemma in combination with the Bolzano-Weierstrass theorem: if (x_n) is a Cauchy sequence of $\mathbb{R}^n$, it is bounded, so by Bolzano-Weierstrass (x_n) has a converging subsequence. Since (x_n) is Cauchy and has a converging subsequence, it converges.

Theorem 5.1. *The metric space $\mathbb{R}^n$ (with the Euclidian distance) is complete.*

6 Continuity

So far we defined the notion of limit for sequences, i.e. functions from $\mathbb{N}$ to a metric space. We now extend the notion of limit to functions from a metric space to another metric space. In the whole section, we note (X, d_X) the departure metric space, (Y, d_Y) the arrival metric space, S a subset of X, and $f : S \to Y$ the function. (Note that f does not need to be defined on the whole space X).

6.1 Limits of functions

A sequence (x_n) converges to l in $+\infty$ iff when n gets closer and closer to $+\infty$, x_n gets closer and closer to l. Similarly, a function f converges to a limit l at a point $x_0 \in X$ iff when x gets closer and closer to x_0, $f(x)$ gets closer and closer to l.

Definition 6.1. *Let $x_0 \in X$ and $f : S \to Y$ a function defined in a neighborhood of x_0 (but not necessarily in x_0) and $l \in Y$. We say that **the limit of** f **at** x_0 **is** l or that f **tends to** l **at** x_0 iff:*

$$\forall \varepsilon > 0, \exists \delta > 0, \big[0 < d(x, x_0) < \delta \Rightarrow d(f(x), l) < \varepsilon\big]$$

We note $f(x) \underset{x \to x_0}{\to} l$ or $\lim_{x \to x_0} f = l$.

The condition $0 < d(x, x_0)$ is equivalent to $x \neq x_0$: we do not care about what is happening is x_0, only in its neighborhood (again, we allow for f not even to be defined in x_0). Note that the definition can be phrased:

$$\forall \varepsilon > 0, \exists \delta > 0, \big[x \in B(x_0, \delta) - \{x_0\} \Rightarrow f(x) \in B(l, \varepsilon)\big]$$
$$\forall \varepsilon > 0, \exists \delta > 0, f(B(x_0, \delta) - \{x_0\}) \subseteq B(l, \varepsilon)$$
$$\forall \varepsilon > 0, \exists \delta > 0, B(x_0, \delta) - \{x_0\} \subseteq f^{-1}(B(l, \varepsilon))$$

Limits of functions are closely connected to limits of sequences, as stressed by the following characterization:

Proposition 6.1. $\lim_{x \to x_0} f = l$ *iff for all sequences (x_n) such that $x_n \neq x_0$ for all n and that tend to x_0, $(f(x_n))$ tends to l.*

Proof. The proof is left as an exercise. □

(We exclude the sequences such that $x_n = x_0$ for some n because we excluded $x = x_0$ from the definition of the limit if a function). Using this characterization, properties derived for limits of sequences transfer painlessly to limits of functions. In particular, the limit is unique and operations on limits of functions follow the same rules as operations on limits of sequences.

6.2 Continuity

Even when f is defined in x_0, does it follow that the limit at x_0 is necessarily $f(x_0)$? No; when f is defined at x_0 and the limit at x_0 is $f(x_0)$, we say that f is continuous in x_0.

Definition 6.2. *Let $x_0 \in S$ (i.e. f is defined in x_0). f is **continuous in** x_0 iff f has a limit at x_0 and this limit is $f(x_0)$, i.e. iff:*

$$\forall \varepsilon > 0, \exists \delta > 0 / d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$$

*The function f is **continuous** iff it is continuous at all points of its domain S.*

The sequential characterization of the limit at one point still applies to continuity at a point:

Proposition 6.2. *f is continuous at x_0 iff for all sequences (x_n) that tend to x_0, $(f(x_n))$ tends to $f(x_0)$.*

(Note that we no longer have to make sure the sequence (x_n) is such that $x_n \neq 0$ for all n). But there is another characterization for continuity on the whole domain:

Theorem 6.1.

- *f is continuous iff the inverse image by f of any open set is an open set.*
- *f is continuous iff the inverse image by f of any closed set is a closed set.*

Proof. Start with the characterization with open sets. Assume f is continuous. Let V be an open set of (Y, d_Y). We want to show that $f^{-1}(V)$ is an open set. Take $x_0 \in f^{-1}(V)$; so $f(x_0) \in V$. Since V is open, there exists an open ball $B(f(x_0), \varepsilon) \subseteq V$. Since f is continuous, there exists an open ball $B(x_0, \delta)$ such that $B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon)) \subseteq f^{-1}(V)$. QED.

Conversely, assume the inverse image of any open set by f is an open set. Fix x_0; we want to show that f is continuous at x_0. Fix $\varepsilon > 0$. Consider the open ball $B(f(x_0), \varepsilon)$. Since it is an open set, by assumption $f^{-1}(B(f(x_0), \varepsilon))$ is open. Since it is open, and $x_0 \in f^{-1}(B(f(x_0), \varepsilon))$, there exists an open ball $B(x_0, \delta)$ such that $B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon))$. QED.

The characterization with closed sets follows directly since $(f^{-1}(S))^c = f^{-1}(S^c)$ and S is closed iff S^c is open. □

A result on composite functions.

Proposition 6.3. *Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces and $f : X \to Y$ and $g : Y \to Z$ be functions. Assume f is defined on $S \subseteq X$ and g is defined on $T \subseteq Y$ such that $f(S) \subseteq T$, so that $g \circ f$ is defined.*

If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Proof. See problem-set. □

6.3 Continuity and Compactness

From the previous subsection, the *inverse image* of a closed set by a continuous function is a closed set. In contrast, the *image* of a closed set by a continuous function needs not be a closed set. Take as a counter-example the function $f(x) = 1/x$ from $\mathbb{R}$ to $\mathbb{R}$, and the set $[1, +\infty)$, which is closed in $\mathbb{R}$: $f([1, +\infty) = (0, 1]$, which is not closed in $\mathbb{R}$. However, the image of a *compact* set by a continuous function is necessarily compact (hence closed):

Theorem 6.2. *Let K be a compact subset of S and f a continuous function. Then $f(K)$ is compact.*

Proof. Let (y_n) be a sequence of $f(K)$. By definition of the image, for all y_n, there exists $x_n \in K$ such that $y_n = f(x_n)$. Consider the sequence (x_n) of K; since K is compact, it has a subsequence (x_{n_k}) that converges to a limit $l \in K$. Consider the subsequence $(y_{n_k}) = (f(x_{n_k}))$ of (y_n). Since f is continuous, it converges to $f(l) \in f(K)$. So (y_n) has a subsequence that converges to a limit in $f(K)$. QED. □

A central application of this theorem is the existence of a maximum and minimum of a function which takes values in $\mathbb{R}$ (endowed with the absolute value metric). First notice that:

Proposition 6.4. *Let S be a bounded subset of $\mathbb{R}$ (endowed with the absolute value metric), so that the supremum and infimum of S exist. Then the supremum $sup(S)$ and infimum $inf(S)$ of S are in the closure of S.*

Proof. Let $r > 0$ and consider the open ball $B(sup(S), r)$ around $sup(S)$. By definition of $sup(S)$, $sup(S) - r$ is not an upper bound of S, so there exists $x \in S$, such that $sup(S) - r < x \leq sup(S)$. This implies that $x \in B(sup(S), r)$: $B(sup(S), r)$ contains an element of S. QED. □

Therefore, to show that a function from a metric space (X, d_X) to $\mathbb{R}$ has a minimum and maximum, it is sufficient to show that $f(X)$ is bounded and closed, i.e. that $f(X)$ is compact. But this is guaranteed if X is compact so:

Theorem 6.3. ***Weierstrass theorem*** *Let f be a function from a metric space (X, d_X) to $\mathbb{R}$.*

If f is continuous and X is compact, then f has a maximum and a minimum.

7 Contraction Mapping Theorem

Many equilibrium notions in economics can be seen as fixed points problems—problems of finding fixed points to a given function:

Definition 7.1. *Let X be a set and $f : X \to X$ a function of X into itself. $x^* \in X$ is a* ***fixed point of*** *f iff $f(x^*) = x^*$.*

This is why fixed point theorems—like the contraction mapping theorem, but not only—are so useful in economics to show the existence of equilibria. The contraction mapping theorem, or Banach fixed point theorem, is one such fixed-point theorem which is also at the heart of many results in dynamic programming. It first requires f to be a contraction:

Definition 7.2. *Let (X, d) be a metric space. A function $f : X \to X$ of X into itself is a* ***contraction mapping*** *or* ***contraction*** *with* ***modulus*** *$k \in [0, 1)$ iff:*

$$\exists k \in [0, 1), \forall x, y \in X, d(f(x), f(y)) \leq kd(x, y)$$

Intuitively, the distance between x and y contracts when we apply the function f to both points. We are ready to state the contraction mapping theorem:

Theorem 7.1. ***Contraction mapping theorem or Banach fixed point theorem***

Let (X, d) be a metric space and $f : X \to X$ a mapping from X to itself.

If (X, d) is complete and f is a contraction mapping, then f has a unique fixed point x^*.

Furthermore, for any element $x \in X$, the sequence (x_n) defined recursively by $x_0 = x$ and $x_n = f(x_{n-1})$ converges to x^*.

The last part of the theorem provides a way to find the fixed point of a contraction numerically.

Proof. See problem set. □