Correspondences

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Simply put, a correspondence is a function that is allowed to take multiple values instead of a single one. Correspondences are given much emphasis in economics (more than in mathematical analysis classes), due to two of their applications. The first application is in optimization: an essential correspondence in optimization is the one that, to each value of the parameter of a parameterized problem, associates the set of maximizers. The second application consists in a fixed point theorem for correspondences—the Kakutani theorem. John Nash used the Kakutani theorem in 1950 to prove the existence of a Nash equilibrium in a large class of games. Arrow and Debreu used the theorem in 1954 to prove the existence of a general equilibrium under the assumption of an Arrow-Debreu economy.

1 Definition

Definition 1.1. *Let X and Y be two sets. A* ***correspondence*** *or* ***set-valued function*** *or* ***multi-valued function*** *Γ from X to Y associates to every element $x \in X$ a subset of Y, noted $\Gamma(x)$. We note:*

$$\Gamma : X \rightrightarrows Y$$
$$x \mapsto \Gamma(x)$$

We call the set $\{(x, y)/x \in X, y \in \Gamma(x)\}$ the ***graph*** *of Γ.*

We can see correspondences as a generalization of functions, since a function from X to Y can be seen as a particular correspondence where $\Gamma(x)$ is a singleton for all x: a single-valued correspondence. But note that a correspondence is actually a function, from X to the power set of Y, $P(Y)$.

Definition 1.2. *Let $\Gamma : X \rightrightarrows Y$ be a correspondence.*

For any subset $S \subseteq X$, the ***image*** *of S under Γ is the set $\Gamma(S) = \bigcup_{x \in S} \Gamma(x)$.*

We say that a correspondence is:

- **non-empty-valued** if for all x, $\Gamma(x) \neq \emptyset$
- **convex-valued** if for all x, $\Gamma(x)$ is convex

If (Y, d_Y) is a metric space, we can define topological properties:

- **closed-valued** if for all x, $\Gamma(x)$ is closed
- **compact-valued** if for all x, $\Gamma(x)$ is compact

2 Upper and Lower Hemicontinuity (and Closed-graph Property)

We now turn to generalizing the notion of continuity from functions to correspondences. To do so, we now assume that both (X, d_X) and (Y, d_Y) are metric spaces.

2.1 Definitions

Simply put, a function f is continuous at x_0 if $f(x)$ is close to $f(x_0)$ when x is close to x_0. More rigorously:

For all open set V such that $f(x_0) \in V$, there exists an open set $U \ni x_0$ st. $\forall x \in U, f(x) \in V$

(The course defined continuity using open balls $B(f(x_0), \varepsilon)$ and $B(x_0, \delta)$, not open sets U and V; it is not difficult to see that the two definitions are equivalent). To adapt the definition to correspondences, we basically want to say that $\Gamma(x)$ is *"close to"* $\Gamma(x_0)$ when x is close to x_0. But what does it mean for two sets to be close? We are actually going to define two notions: that $\Gamma(x)$ cannot be much bigger than $\Gamma(x_0)$ when x is close to x_0—upper hemicontinuity; and that $\Gamma(x)$ cannot be much smaller than $\Gamma(x_0)$ when x is close to x_0—lower hemicontinuity.

First a correspondence Γ is upper hemicontinuous at x if $\Gamma(x)$ does not "explode" into a set much larger than $\Gamma(x_0)$ when moving away from x_0: if $\Gamma(x_0)$ is contained in an open set V, so does $\Gamma(x)$ for x close to x_0. Formally, we replace $f(.) \in V$ by $\Gamma(.) \subseteq V$ in the definition of continuity for functions.

Definition 2.1. *Let $\Gamma : (X, d_X) \rightrightarrows (Y, d_Y)$ be a correspondence.*
*Γ is **upper hemicontinuous (uhc) at** x_0 iff:*

For all open set V such that $\Gamma(x_0) \subseteq V$, there exists an open set $U \ni x_0$ st. $\forall x \in U, \Gamma(x) \subseteq V$

*Γ is **upper hemicontinuous (uhc)** iff it is upper hemicontinuous at all $x \in X$.*

Second, a correspondence Γ is lower hemicontinuous at x if $\Gamma(x)$ does not "shrink" into a much smaller set than $\Gamma(x_0)$ when moving away from x_0: if $\Gamma(x_0)$ intersects an open set V, so does $\Gamma(x)$ for x close to x_0. Formally, we replace $f(.) \in V$ by $V \cap \Gamma(.) \neq \emptyset$ in the definition of continuity for functions.

Definition 2.2. *Let $\Gamma : (X, d_X) \rightrightarrows (Y, d_Y)$ be a correspondence.*
*Γ is **lower hemicontinuous (lhc) at** x_0 iff:*

For all open set V such that $V \cap \Gamma(x_0) \neq \emptyset$, there exists an open set $U \ni x_0$ st. $\forall x \in U, V \cap \Gamma(x) \neq \emptyset$

*Γ is **lower hemicontinuous (lhc)** iff it is lower hemicontinuous at all $x \in X$.*

The two hemi-continuities define continuity:

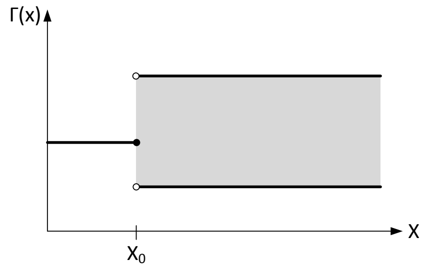
Definition 2.3. *A correspondence is **continuous** (at x_0) if it is both uhc and lhc (at x_0).*

For a single-valued correspondence f—a function—both $V \subseteq f(.)$ and $V \cap f(.) \neq \emptyset$ correspond to $f(.) \in V$, so the definitions of both upper and lower hemi-continutiy correspond to the definition of continuity for functions.

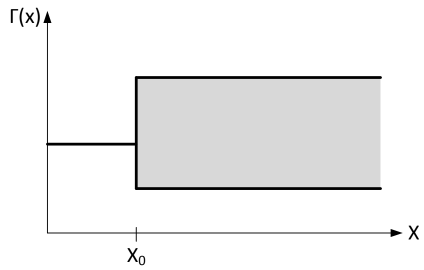
Proposition 2.1.

- *A single-valued correspondence is uhc iff it is continuous as a function.*
- *A single-valued correspondence is lhc iff it is continuous as a function.*

(*"As a function"* here means for the notion of continuity defined for functions). The figure below gives a graphical illustration of the "no explosion" and "no shrinkage" intuitions behind the notions of uhc and lhc.



Γ(x) explodes when moving away from x_0 to the right: Γ is not uhc.



Γ(x) shrinks when moving away from x_0 to the left: Γ is not lhc.

2.2 Sequential characterization of lhc

Lower hemicontinuity is most often used with its sequential characterization:

Proposition 2.2. *Let* $\Gamma : (X, d_X) \rightrightarrows (Y, d_Y)$ *be a correspondence.*

Γ *is* ***lower hemicontinuous (lhc) at*** x_0 *iff: for any sequence* $(x_n) \in X^{\mathbb{N}}$ *that converges to* x_0*, and for any* $y \in \Gamma(x_0)$*, there exists a sequence* $(y_n) \in Y^{\mathbb{N}}$ *such that* $y_n \in \Gamma(x_n)$ *(at least after a certain term* $n \geq N$*), and* (y_n) *converges to* y*.*

Proof. Assume Γ is lhc at x_0. Fix a sequence $x_n \to x_0$ and fix $y \in \Gamma(x_0)$. We want to build a sequence (y_n) such that $y_n \in \Gamma(x_n)$ for all n and $y_n \to y$. Fix an integer k. Choose the open set $V = B(y, \frac{1}{k})$ in the definition of lhc: since $\Gamma(x_0) \cap B(y, \frac{1}{k}) \neq \emptyset$ (it contains y!), there exists an open ball $B(x_0, \varepsilon)$ such that for all $x \in B(x_0, \varepsilon)$, $\Gamma(x) \cap B(y, \frac{1}{k}) \neq \emptyset$. Now use the definition of $x_n \to x_0$: there exists N_k such that for all $n \geq N_k$, $x_n \in B(x_0, \varepsilon)$, hence $\Gamma(x_n) \cap B(y, \frac{1}{k}) \neq \emptyset$. Now to define y_n for some n, pick the greatest k such that $N_k \leq n$, and take as y_n an element of $\Gamma(x_n) \cap B(y, \frac{1}{k}) \neq \emptyset$. This guarantees that $y_n \in \Gamma(x_n)$. We just need to prove that $y_n \to y$. Fix $\varepsilon > 0$; there exists k such that $\frac{1}{k} < \varepsilon$. So there exists N_k such that for all $n \geq N_k$, $y_n \in B(y, \frac{1}{k}) \subseteq B(y, \varepsilon)$.

Conversely, show the contrapositive. Assume Γ is not lhc at x_0: there exists an open set V such that $V \cap \Gamma(x_0) \neq \emptyset$ but for all $U \ni x_0$, there exists $x \in U$ such that $V \cap \Gamma(x) = \emptyset$. Because $V \cap \Gamma(x_0)$ is not empty, there exists $y \in V \cap \Gamma(x_0)$. We now build a sequence (x_n). Fix n; $B(x_0, 1/n)$ is an open set U containing x_0, so there exists some element in $B(x_0, 1/n)$, that we define as x_n, such that $\Gamma(x_n) \cap V = \emptyset$. This way, we build a sequence (x_n) such that $x_n \to x_0$ and for all n, $\Gamma(x_n) \cap V = \emptyset$. To conclude, we need to show that for any sequence (y_n) such that $y_n \in \Gamma(x_n)$ for all n, (y_n) does not converge to y. But since V is open, there exists ε such that $B(y, \varepsilon) \subseteq V$. We have that for all n, $\Gamma(x_n) \cap B(y, \varepsilon) = \emptyset$, hence $y_n \notin B(y, \varepsilon)$. Since for all n, $y_n \notin B(y, \varepsilon)$, (y_n) cannot converge to y. QED. □

2.3 Sequential characterization of uhc and the closed-graph property

2.3.1 Sequential sufficient condition for uhc

Upper hemicontinuity is easier to use with the open sets definition above, but there exists a sufficient condition for upper hemicontinuity expressed with sequences, which turns into a sequential characterization if the correspondence is compact-valued.

Proposition 2.3. *Let* $\Gamma : (X, d_X) \rightrightarrows (Y, d_Y)$ *be a correspondence.*

If for any sequence $(x_n) \in X^{\mathbb{N}}$ *that converges to* x_0*, and for any sequence* $(y_n) \in Y^{\mathbb{N}}$ *such that* $y_n \in$

$\Gamma(x_n)$ for all n, there exists a subsequence of (y_n) that converges to a points in $\Gamma(x_0)$, then Γ is upper-hemicontinuous at x_0.

If Γ is compact-valued, then the converse is also true.

Proof. You are asked to prove the sufficient condition in the problem set. We *admit* the partial converse. □

2.3.2 The closed-graph property and its connection to uhc

We define a third property, which does not generalize continuity, but is connected to upper hemicontinuity, as shown below.

Definition 2.4. *Let $\Gamma : (X, d_X) \rightrightarrows (Y, d_Y)$ be a correspondence.*

*Γ has the **closed-graph property at** x_0 iff for any sequence $(x_n) \in X^{\mathbb{N}}$ that converges to x_0, and for any $(y_n) \in Y^{\mathbb{N}}$ such that $y_n \in \Gamma(x_n)$ for all n, if (y_n) converges to a limit y, then $y \in \Gamma(x_0)$.*

*Γ has the **closed-graph property** iff it has the closed-graph property at all $x \in X$.*

The following results are easy to derive:

- A correspondence has the closed-graph property (at all $x \in X$) iff its graph is closed.

  (We implicitly consider the **product metric space** $(X \times Y, d_{X\times Y})$ endowed with the **product metric** $d_{X\times Y}((x,y),(x',y')) = \max(d_X(x,x'), d_Y(y,y'))$. With this metric, a sequence (x_n, y_n) converges to (x, y) iff (x_n) converges to x and (y_n) converges to y. The result is just the sequential definition of closeness).

- If Γ has the closed-graph property, then it is closed-valued.

  (Use the sequential definition of closeness and a sequence (x_n) constant to x_0).

- But if Γ is closed-valued, it needs not have the closed-graph property.

  (As a counter-example: $\Gamma(x) = [0,1]$ for $x \leq 0$ and $\Gamma(x) = [0,2]$ for $x > 0$).

Contrary to upper and lower hemicontinuity, the closed-graph property does not generalize continuity of functions: a single-valued correspondence may have the closed-graph property at x but not be continuous at x if looked at as a function. Indeed, if f is a function:

- f is continuous at x iff for any sequence $x_n \to x$, the sequence $f(x_n) \to f(x)$.

- f has the closed-graph property at x iff for any sequence $x_n \to x$, <u>if the sequence $(f(x_n))$ converges</u>, then $f(x_n) \to f(x)$.

It is therefore easy to build an example of a function that has the closed-graph property but is not continuous: for instance, consider $f(x) = 0$ for $x \leq 0$ and $f(x) = 1/x$ for $x > 0$ at $x = 0$.

Now, how do uhc and the closed-graph property connect? Although:

- Upper hemicontinuity does not imply the closed-graph property

  (Consider the constant correspondence over $\mathbb{R}$, $\Gamma(x) =]0,1[$, which is uhc but not closed-valued, hence not closed-graph).

- The closed-graph property does not imply upper hemicontinuity

  (Consider again $f(x) = 0$ for $x \leq 0$ and $f(x) = 1/x$ for $x > 0$ at $x = 0$, which is closed-graph, but discountinuous, hence not uhc).

there exist connections between the two notions if we strengthen the assumptions.

Proposition 2.4. *Let* $\Gamma : (X, d_X) \rightrightarrows (Y, d_Y)$ *be a correspondence.*

If Γ *is closed-valued and uhc, then is it closed-graph.*

Proof. We admit the result. □

To go from closed-graph to uhc, we strengthen the assumption with local boundedness:

Definition 2.5. *Let* $\Gamma : (X, d_X) \rightrightarrows (Y, d_Y)$ *be a correspondence.*

Γ *is* ***locally bounded at*** x *iff there exists an open set* $U \ni x$ *and a compact set* $K \subseteq Y$ *st.* $\Gamma(U) \subseteq K$*.*

Γ *is* ***locally bounded*** *iff it is* ***locally bounded*** *at all* $x \in X$*.*

Note that a sufficient condition for local boundedness (at all $x \in X$) is that Y itself be compact.

Proposition 2.5. *Let* $\Gamma : (X, d_X) \rightrightarrows (Y, d_Y)$ *be a correspondence.*

If Γ *is closed-graph (at* x_0*) and locally bounded (at* x_0*), then is it uhc (at* x_0*).*

Proof. Assume Γ is closed-graph at x_0 and locally bounded at x_0, and show the sufficient condition for uhc at x_0. Consider a sequence (x_n) that converges to x_0, and a sequence (y_n) such that $y_n \in \Gamma(y_n)$ for all n. Since Γ is locally bounded at x_0, (y_n) necessarily lies in the compact set K after a certain rank. So (y_n) has a converging subsequence. But the limit is not necessarily in $\Gamma(x_0)$, so Γ is not necessarily uhc at x_0. This is why we also need Γ to be closed-graph at x_0. □

If Y is compact, then local boundedness is always satisfied (at all x), so closed-graphness always implies uhc. Besides, we have seen that close-graphness implies closed-valuedness. So:

Corollary 2.1. *Let* $\Gamma : (X, d_X) \rightrightarrows (Y, d_Y)$ *be a correspondence.*

Assume Y *is compact. Then:*

$$\Gamma \text{ closed-graph} \Leftrightarrow \Gamma \text{ uhc and closed-valued.}$$

Hence, when Y is compact, we can use the closed-graph sequential definition as a sequential characterization of uhc.

3 The maximum theorem

Consider family of maximization problems parameterized by θ. The maximum theorem gives sufficient conditions on the objective function $f(x, \theta)$ and the correspondence of constraint sets $\mathcal{D}(\theta)$ for the value function and the correspondence of maximizers to be continuous. The main conditions are the continuity of the objective function $f(x, \theta)$ and the constraint set $\mathcal{D}(\theta)$.

Theorem 3.1. ***Maximum Theorem***

Let (X, d_X) and (Θ, d_θ) be metric spaces, $f : X \times \Theta \to \mathbb{R}$ a function, and $\mathcal{D}(\theta) : \Theta \rightrightarrows P(X)$ a correspondence.

Consider the parameterized family of programs: $\max_{x \in \mathcal{D}(\theta)} f(x, \theta)$. Let:

$$f^*(\theta) \equiv \max_{x \in \mathcal{D}(\theta)} f(x, \theta)$$

$$\mathcal{D}^*(\theta) \equiv \{x \in \mathcal{D}(\theta) : f(x, \theta) = f^*(\theta)\}.$$

be the value function and the correspondence of maximizers. If:

1. *f is continuous,*
2. *$\mathcal{D}$ is non-empty-valued, compact-valued and continuous (uhc and lhc),*

then:

1. *f^* is a continuous function,*
2. *$\mathcal{D}^*$ is a non-empty-valued, compact-valued, and upper hemicontinuous correspondence.*

Proof. We *admit* the result. □

Note that $\mathcal{D}^*$ may fail to be lower hemicontinuous.

Corollary 3.1. *If in addition, $\mathcal{D}$ is convex-valued and f is strictly quasi-concave, we know that $\mathcal{D}^*$ is single-valued, so that $\mathcal{D}^*$ is a continuous function.*

4 The Brouwer and Kakutani fixed-point Theorems

The Brouwer and Kakutani theorems are two fixed-point theorems: the Brouwer theorem is a fixed-point theorem for functions, and the Kakutani theorem is an extension to correspondences. Let us start with the Brouwer theorem.

Theorem 4.1. ***Brouwer's fixed point theorem***

Let $f : S \to S$, $S \subseteq \mathbb{R}^n$ be a function. If:

1. *S is non-empty, compact and convex*
2. *f is continuous*

then f has a fixed point.

Proof. We *admit* the result. ☐

Before stating Kakutani theorem, we need to first define what we mean by a fixed point of a correspondence:

Definition 4.1. *A fixed point of Γ is a point $x \in X$ such that $x \in \Gamma(X)$.*

Theorem 4.2. ***Kakutani's fixed point theorem***

Let $S \subseteq \mathbb{R}^n$ and $\Gamma : S \rightrightarrows S$ be a correspondence. If:

1. *S is non-empty, compact and convex*
2. *Γ is non-empty-valued, convex-valued, and closed-graph (closed-valued and UHC)*

then Γ has a fixed point.

Proof. We *admit* the result. ☐